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SPONTANEOUS BREAKDOWN OF THE LORENTZ INVARIANCE IN THREE-DIMENSIONAL GAUGE THEORIES†

YUTAKA HOSOTANI
School of Physics and Astronomy, University of Minnesota, Minneapolis, MN 55455

ABSTRACT

In a class of renormalizable three-dimensional abelian gauge theory the Lorentz invariance is spontaneously broken by dynamical generation of a magnetic field B. The true ground state resembles that of the quantum Hall effect. An originally topologically massive photon becomes gapless, fulfilling the role of the Nambu-Goldstone boson associated with the spontaneous breaking of the Lorentz invariance. We give a simple explanation and a sufficient condition for the spontaneous breaking of the Lorentz invariance with the aid of the Nambu-Goldstone theorem. The decrease of the energy density by $B\neq0$ is understood mostly due to the shift in zero-point energy of photons.

1. Variational Ground State

In a wide class of three-dimensional gauge theories described by[1,2]

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\kappa_0}{2}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \\ &+\sum_a \frac{1}{2}\left[\,\overline{\psi}_a\,,\,(\gamma^\mu_a(i\partial_\mu + q_a A_\mu) - m_a)\psi_a\,\right]\,,\end{aligned} \tag{1}$$

the Lorentz invariance is spontaneously broken by dynamical generation of a magnetic field B. We have constructed a variational ground state which has $B\neq0$ and has a lower energy density than the perturbative vacuum. The theory is renormalizable.

Taking advantage of charge-conjugation invariance of (1), one can take $q_a>0$ without loss of generality. Chirality of a Dirac particle is defined by $\eta_a=\frac{i}{2}\,\mathrm{Tr}\,\gamma^0_a\gamma^1_a\gamma^2_a$ $=\pm1$ (for $m_a\geq0$). It is equivalent to the sign of m_a for $m_a\neq0$ with $\eta_a=+$, as can be understood by making a transformation $\{\gamma^\mu\}\to\{-\gamma^\mu\}$.

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Suppose that a uniform $B \neq 0$ is dynamically generated. The energy spectrum of Dirac particles defines Landau levels:

$$\begin{array}{lll} E_0 = & \epsilon(\eta_a B) \cdot m_a & (n=0) \\ E_n^{\pm} = & \pm(m_a^2 + 2nq_a|B|)^{1/2} & (n \geq 1). \end{array} \tag{2}$$

Each Landau level has the density of states $q_a|B|/2\pi$. There is asymmetry in the $n=0$ modes (zero modes). They exist in either positive or negative energy states, or in other words, only for either particles or anti-particles. In the $m_a \to 0$ limit the lowest energy level $|E_0|$ approaches zero so that the ground state energy does not depend, at least in the leading order in perturbation theory, on whether the lowest level is partially filled or not. In other words the perturbative ground states in the $m_a \to 0$ limit are infinitely degenerate. The degeneracy is lifted by quantum corrections.

We consider variational ground states in which these lowest Landau levels are either empty or completely filled.[3] Accordingly a filling factor $\nu_a = 0$ or 1 is assigned to each species of fermions. A variational ground state is denoted by $\Psi_{\rm g.s.}(B, \{\nu_a\})$. In general one should consider more general filling factors. Among them Laughlin-type states with $\nu = \frac{1}{3}, \frac{1}{5} \cdots$, are good candidates. In this article we confine ourselves to $\nu = 0$ or 1. We shall show that when $\{\nu_a\}$ satisfies a certain condition, the state $\Psi_{\rm g.s.}(B, \{\nu_a\})$ has a lower energy density than the perturbative vacuum with $B = 0$.

2. Consistency Condition

With fixed Dirac matrices one can continuously change the value of fermion mass m_a from positive to negative. Except for zero-modes in the spectrum (2) all positive (negative) frequencies remain positive (negative). However, there results a crossing in zero modes. Positive frequency zero-modes become negative frequency zero-modes, or vice versa. Its implication is that under the continuous change of m_a an empty state $\nu_a = 0$ with positive m_a becomes a completely filled state $\nu_a = 1$ with negative m_a.

To see it, let us suppose $\eta_a B > 0$. For a positive m_a zero modes exist only for positive energy solutions. Hence we have schematically

$$\psi = \sum_k \Big\{ \sum_{n=0}^{\infty} a_{nk} u_{nk}(x) + \sum_{n=1}^{\infty} b_{nk}^\dagger w_{nk}(x) \Big\}$$

where $\{u_{nk}\}$ and $\{w_{nk}\}$ represent positive and negative energy solutions. The

charge is

$$
\begin{aligned}
Q &= \int d^2x \, \frac{q}{2} \, [\psi^\dagger, \psi] \\
&= q \bigg\{ \sum_k (a_{0k}^\dagger a_{0k} - \frac{1}{2}) + \sum_{n=1}^{\infty} \sum_k (a_{nk}^\dagger a_{nk} - b_{nk}^\dagger b_{nk}) \bigg\} \quad .
\end{aligned} \tag{3}
$$

When the mass m_a continuously varies from positive to negative, u_{0k} becomes a negative frequency solution so that an annihilation operator a_{0k} needs to be re-interpreted as a creation operator $b_{0k}^\dagger$ of an anti-particle. In the expression for Q, $a_{0k}^\dagger a_{0k} - \frac{1}{2}$ is transformed into $b_{0k} b_{0k}^\dagger - \frac{1}{2} = \frac{1}{2} - b_{0k}^\dagger b_{0k}$. This implies that $\nu_a = 0$ (1) with $m_a > 0$ becomes $\nu_a = 1$ (0) with $m_a < 0$.

Since fermions with $(\eta_a, m_a < 0)$ are equivalent to those with $(-\eta_a, |m_a|)$, a continuous change $m_a \to -m_a$ results in the transformation of $(\eta_a, \nu_a, |m_a|) \to (-\eta_a, 1 - \nu_a, |m_a|)$. Any physical quantities, R, must satisfy

$$
R(\eta_a, \nu_a, m_a^2) = R(-\eta_a, 1 - \nu_a, m_a^2) \quad . \tag{4}
$$

The charge in the presence of a magnetic field is found from (3). For $m_a > 0$

$$
\begin{aligned}
Q_a &= q_a \, \eta_a \epsilon(B) \sum_k (\nu_a - \tfrac{1}{2}) \\
&= q_a \, \eta_a \epsilon(B) \cdot \frac{q_a |B|}{2\pi} (volume) \cdot (\nu_a - \tfrac{1}{2})
\end{aligned}
$$

where we have recovered the factor $\eta_a \epsilon(B)$. Hence the charge density is found to be

$$
\langle \, j^0(x) \, \rangle = \frac{1}{2\pi} \sum_a \eta_a q_a^2 \, (\nu_a - \tfrac{1}{2}) \cdot B \ , \tag{5}
$$

which satisfies (4).

An important corollary follows. To satisfy the field equation one must have $\kappa_0 B = \langle \, j^0 \, \rangle$. Hence in order to have $B \neq 0$, the relation

$$
\kappa_0 = \frac{1}{2\pi} \sum_a \eta_a q_a^2 \, (\nu_a - \tfrac{1}{2}) \tag{6}
$$

must be satisfied. This is a consistency condition for having $B \neq 0$.

One can integrate Dirac fields in (1) in a background field $\bar{F}_{12} = -B$ with specified filling fractions $\{\nu_a\}$. Let us denote the fluctuation part of A_μ by A'_μ. The resulting effective Lagrangian is summarized as

$$
\begin{aligned}
\mathcal{L}_{\rm eff}[A'] &= -\frac{1}{2} B^2 + B F'_{12} \\
&\quad + \frac{1}{2} \, F'_{0k} \, \epsilon \, F'_{0k} - \frac{1}{2} \, F'_{12} \, \chi \, F'_{12} - \frac{1}{2} \, \varepsilon^{\mu\nu\rho} A'_\mu \, \kappa \, \partial_\nu A'_\rho + {\rm O}(A'^3)
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
\epsilon &= 1+\Pi_0(p) \\
\chi &= 1+\Pi_2(p) \\
\kappa &= \kappa_0-\Pi_1(p) \qquad\qquad (p_\mu = i\partial_\mu) \ .
\end{aligned}
\tag{8}
$$

Here $\Pi_0(p)$, $\Pi_1(p)$, and $\Pi_2(p)$ represent fermion one-loop effects, which depend on B and $\{\nu_a\}$. In particular, $\Pi_1(0)=(2\pi)^{-1}\sum_a \eta_a q_a^2\,(\nu_a-\frac{1}{2})$. $-\Pi_1(0)$ is the induced Chern-Simons coefficient. In the perturbative vacuum[4] $\Pi_1(0)_{\rm p.v.}=-\sum_a \eta_a q_a^2/4\pi$. The consistency condition (6) can be written as

$$
B\cdot\kappa(0)=0 \ \ . \tag{9}
$$

To have $B\neq 0$, the total Chern-Simons coefficient at zero momentum must vanish.

3. Energy Density

Whether or not a non-vanishing B is dynamically generated is determined by examining the difference in the energy densities of the variational ground state and perturbative vacuum, $\Delta\mathcal{E}=\mathcal{E}_{\rm g.s.}(B,\{\nu_a\})-\mathcal{E}_{\rm p.v.}$. It is found to be

$$
\begin{aligned}
\Delta\mathcal{E} =& \ \frac{1}{2}\,B^2+\Delta\mathcal{E}_{\rm f} \\
&+\int_0^1 \frac{d\alpha}{\alpha} \ \ i\int\frac{d^3p}{(2\pi)^3} \ \ {\rm tr}\,D_0^{-1}(p)\Big\{D_{\rm g.s.}(p\ ;B,\{\nu_a\})-D_{\rm p.v.}(p)\Big\}
\end{aligned}
\tag{10}
$$

The second term, $\Delta\mathcal{E}_{\rm f}$, represents the shift in fermion zero-point energies. In the last term we have introduced an auxiriary parameter α representing the coupling of a fluctuation part A'_μ of gauge fields to fermions. Full gauge field propagators $D_{\rm g.s.}(p)$ and $D_{\rm p.v.}(p)$ are determined with a given α. The formula (10) is exact, involving no approximation.

We evaluate the gauge field propagators in the random phase approximation in which an infinite series of ring diagrams are summed. It is equivalent to keep terms in (6) up to $O(A'^2)$ to determine the propagators. Then the α-integration in (10) can be easily performed. The result is

$$
\begin{aligned}
&\Delta\mathcal{E}=\frac{1}{2}B^2+\Delta\mathcal{E}_{\rm f}-\frac{i}{2}\int\frac{d^3p}{(2\pi)^3} \ \ln\frac{\mathcal{S}\,(p)_{\rm g.s.}}{\mathcal{S}\,(p)_{\rm p.v.}} \\
&\mathcal{S}\,(p)=\epsilon^2p_0^2-\epsilon\chi\vec{p}^{\,2}-\kappa^2 \ .
\end{aligned}
\tag{11}
$$

Let us consider a chirally symmetric model consisting of $N_{\rm f}$ pairs of $\eta_a=+$ and $-$ fermions with the same mass $m_a\geq 0$ and charge $q_a>0$. In this model one has

$\sum_a \eta_a q_a^2 = 0$, and $\Pi_1(p)_{\rm p.v.} = 0$ exactly in the perturbative vacuum. We suppose that the bare Chern-Simons coefficient and filling factors $\{\nu_a\}$ of the variational ground state satisfy the condition (6): $\kappa_0 = \sum_a \eta_a \nu_a q_a^2/2\pi \neq 0$.

In the massless fermion limit

$$\begin{aligned} \Delta\mathcal{E} = \frac{1}{2}B^2 + \sum_a \frac{\zeta(\frac{3}{2})}{2^{5/2}\pi^2} \, |q_a B|^{3/2} \\ - \frac{\sum \eta_a \nu_a q_a^3}{2\pi^3} \cdot \tan^{-1} \frac{8 \sum \eta_a \nu_a q_a^2}{\pi \sum q_a^2} \cdot |B| + {\rm O}(|B|^{3/2}). \end{aligned} \tag{12}$$

A wide class of models give a negative coefficient for the linear term ($\propto |B|$). As an example, suppose that $\nu_a = 1$ ($\nu_a = 0$) for $\eta_a = +$ ($\eta_a = -$), and $\kappa_0 = \sum_{\eta=+} q_a^2/2\pi$. In this case (12) becomes

$$\Delta\mathcal{E} = -\frac{\sum q_a^3}{4\pi^3} \cdot \tan^{-1} \frac{4}{\pi} \cdot |B| + {\rm O}(|B|^{3/2}) \ . \tag{13}$$

Hence the energy density is minimized at $B \neq 0$ so that the Lorentz invariance is spontaneously broken.[5]

4. Zero-Point Energies

There is a simple way to understand how the linear term ($\propto |B|$) appears in the energy density (12) and why its coefficient turns out negative.[2] The relevant ingredient is the shift in zero-point energies of photons.

In perturbation theory a photon is topologically massive, with a mass given by the bare Chern-Simons coefficient $|\kappa_0|$. In our ground state $\Psi_{\rm g.s.}(B, \{\nu_a\})$ the photon spectrum is determined by

$$\mathcal{S}\,(p) = \epsilon^2 p_0^2 - \epsilon\chi \vec{p}^{\,2} - \kappa^2 = 0 \quad . \tag{14}$$

The gapful or gapless nature of the spectrum is controled by the total Chern-Simons coefficient $\kappa(p)$. We have seen above that $\kappa(0) = 0$ in $\Psi_{\rm g.s.}$. In other words, a photon becomes gapless.

If $\kappa_0 \neq 0$, the photon spectrum changes significantly, which causes the change in zero-point energies. It is estimated in the following manner.

In $\Psi_{\rm g.s.}$, $\kappa(p)$ vanishes at $p = 0$ but approaches the perturbative value κ_0 at large p. The crossover is found to take place at a scale of the "average" inverse magnetic length $l_{\rm ave}^{-1}$:

$$\frac{1}{l_{\rm ave}^2} = \left| \frac{\sum \eta_a \nu_a q_a^2 / l_a^2}{\sum \eta_a \nu_a q_a^2} \right| = \frac{|\sum \eta_a \nu_a q_a^3|}{|\sum \eta_a \nu_a q_a^2|} \cdot |B| \ . \tag{15}$$

If one suppresses the effect of $\epsilon(p)$ and $\chi(p)$, the photon spectrum is approximately given by $p_0 \sim (\vec{p}^2 + \kappa^2)^{1/2}$. Hence the shift in zero-point energies for small $|B|$ is estimated as

$$
\begin{aligned}
\Delta\mathcal{E}_{\rm z.p.} &\sim \int^{l_{\rm ave}^{-1}} \frac{d\vec{p}}{(2\pi)^2}\, \frac{1}{2}\bigg\{ \sqrt{\vec{p}^{\,2}} - \sqrt{\vec{p}^{\,2}+\kappa_0^2} \bigg\} \\
&= -\frac{1}{8\pi}\frac{|\kappa_0|}{l_{\rm ave}^2} + {\rm O}\Big(\frac{1}{l_{\rm ave}^3}\Big) \\
&= -\frac{|\sum \eta_a \nu_a q_a^3|}{16\pi^2} \cdot |B| + {\rm O}(|B|^{3/2}) ~.
\end{aligned} \tag{16}
$$

Comparing (12) and (16), one finds that in a typical model the shift in zero-point energies of photons explains about 50% of the effect.

The crucial fact is that a photon is originally gapful, but becomes gapless in the new ground state. The linearity in $|B|$ is special to two spatial dimensions.

5. Nambu-Goldstone Theorem

The nonvanishing $|B|$ implies that the Lorentz invariance is spontaneously broken, which leads to the existence of the Nambu-Goldstone boson. The associated Ward identities are

$$
\lim_{p\to 0} p_\rho \, {\rm FT} \langle\, {\rm T}[\mathcal{M}^{0j\rho} F_{0k}] \,\rangle = -\epsilon^{jk} \langle\, F_{12} \,\rangle = \epsilon^{jk} B ~~, \tag{17}
$$

where $\mathcal{M}^{\mu\nu\rho}(x)$ is the angular momentum density and FT stands for a Fourier transform. We have made use of $\partial_\sigma \langle\, F_{\mu\nu}(x) \,\rangle = 0$. Nonvanishing B on the r.h.s. implies that there must be gapless poles in the correlation function $\langle\, \mathcal{M}F \,\rangle$ on the l.h.s..

Since we have two identities, namely for $j=1,2$, naively one might expect two Nambu-Goldstone bosons, corresponding to two broken Lorentz-boost generators. However, there seems to appear only one Nambu-Goldstone boson which couples to both broken generators. A photon is the Nambu-Goldstone boson, and it seems to saturate both Ward identities.[6,7]

It is straightforward to see that a photon is the Nambu-Goldstone boson associated with the spontaneous breaking of the Lorentz invariance. First notice that in terms of symmetric energy-momentum tensors $\Theta^{\mu\nu}$, $\mathcal{M}^{\mu\nu\rho} = x^\mu \Theta^{\nu\rho} - x^\nu \Theta^{\mu\rho}$. The Ward identities (17) become

$$
\lim_{p\to 0} i p_\rho \bigg\{ \frac{\partial}{\partial p_0} {\rm FT} \langle\, {\rm T}[\Theta^{j\rho} F_{0k}] \,\rangle - \frac{\partial}{\partial p_j} {\rm FT} \langle\, {\rm T}[\Theta^{0\rho} F_{0k}] \,\rangle \bigg\} = \epsilon^{jk} B ~~. \tag{18}
$$

Further we recall that

$$\begin{aligned}\Theta^{\mu\nu} =& - F^{\mu\lambda} F^{\nu}{}_{\lambda} + g^{\mu\nu} \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} \\ &+ \sum_a \frac{i}{4} \Big\{ \overline{\psi}_a (\gamma^\mu D^\nu + \gamma^\nu D^\mu) \psi_a - \overline{\psi}_a (\gamma^\mu \overleftarrow{D}^\nu + \gamma^\nu \overleftarrow{D}^\mu) \psi_a \Big\}\end{aligned} \tag{19}$$

In $\Psi_{\rm g.s.}$, $\langle F_{12} \rangle = -B$ and $\langle j^0 \rangle \propto B$. Hence $\Theta^{\mu\nu}$ yields terms of the form $B \cdot F'_{\rho\sigma}$ or $B \cdot j^\rho$, which give non-vanishing contributions in (18). A photon couples to both $F'^{\rho\sigma}$ and j^ρ. It is yet to be seen if the photon pole saturates the Ward identities (18).

As mentioned in the previous section, the gapless nature of a photon also follows from the consistency condition (9), or $\kappa(0)=0$. The condition $\kappa(0)=0$ is vital to have non-vanishing contributions on the l.h.s. of (18) in the $p \to 0$ limit.

We conclude that a photon is the Nambu-Goldstone boson associated with the spontaneous breakdown of the Lorentz invariance.

6. Mechanism At Work

Now one can understand why and how the spontaneous breakdown of the Lorentz invariance takes place in the model. We assume $\kappa_0 \neq 0$ so that a photon is originally topologically massive.

Suppose that $B \neq 0$ is dynamically generated. Then it implies that the Lorentz symmetry is spontaneously broken. As shown above, a photon is the Nambu-Goldstone boson. It becomes gapless. This change of the spectrum causes a shift in zero-point energies, lowering the energy density as seen in sections 3 and 4. Hence a non-vanishing B is prefered energetically. The initial ansatz is justified. The mechanism works in a self-consistent cycle.

With this peception one recognizes that the consisitency condition (6) or (9) is viewed practically as a sufficient condition for the spontaneous breaking of the Lorentz invariance. If $\kappa_0 \neq 0$ and a set $\{\nu_a\}$ satisfying (6) can be found, then $B \neq 0$ is consistently realized with a lower energy density.

One can relax the condition and consider more general models. It is interesting to investigate chirally asymmetric models, models with heavy fermions, or models with a general value of κ_0. How about extensions to higher dimensions? Can the mechanism induce spontaneous compactification in higher dimensional theories such as string theory? Many problems are left for further study.

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